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\mathcal{L}_p -EQUIVALENCE OF IMPULSE DIFFERENTIAL EQUATIONS

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Abstract. By the Schauder's fixed point theorem the \mathcal{L}_p -equivalence between two impulse differential equations is proved.

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1. Introduction

Impulse differential equations constitute a useful mathematical apparatus for the investigation of evolutionary processes in physics, chemistry, control theory and robotics which are subject to the action of short-time forces in the form of impulses. The work of Mil'man and Myshkis [1] marked the beginning of the mathematical theory of these equations.

In the present paper an \mathcal{L}_p -equivalence between two arbitrary impulse differential equations is proved. That means that for every solution of the first equation there exists a solution of the second equation such that the difference of these solutions lies in the space \mathcal{L}_p , and vice versa. Similar problems under other conditions are considered in [2], [3].

2. Statement of the problem

Let $X = \mathbb{R}^n$ be the n -dimensional Euclidean space with identity operator I and norm $\|\cdot\|$. By $T = \{t_n\}$ we denote a sequence of points $0 = t_0 < t_1 < t_2 < \dots$ satisfying the condition $\lim_{n \rightarrow \infty} t_n = \infty$.

Consider the impulse equation

$$\frac{dx}{dt} = F(t, x) \quad \text{for } t \neq t_n \quad (1)$$

$$x(t_n^+) = Q_n x(t_n) \quad \text{for } t = t_n \quad (2)$$

and

$$\frac{dy}{dt} = G(t, y) \quad \text{for } t \neq t_n \quad (3)$$

$$y(t_n^+) = D_n y(t_n) \quad \text{for } t = t_n, \quad (4)$$

where $F(t, x), G(t, y) : \mathbb{R}_+ \times X \rightarrow X$ ($\mathbb{R}_+ = [0, \infty)$) are continuous functions and $Q_n, D_n : X \rightarrow X$ ($n = 1, 2, \dots$). Moreover, we assume that all considered functions are continuous from the left.

Definition 1. We shall say that the function $\psi(t)$ ($t \geq 0$) is a solution of the equation (1) – (2) ((3) – (4)) if for $t \neq t_n$, it satisfies equation (1) ((3)) and for $t = t_n$ the condition of “jump” (2) ((4)).

Let $1 \leq p < \infty$. By B_r we denote the closed ball in the space X with a center at zero and radius r .

Let $\Omega \subset \mathbb{R}_+$. By $\mathfrak{L}_p(\Omega, X)$ we denote the space of all functions $x : \Omega \rightarrow X$ for which $\int_{\Omega} \|x(t)\|^p dt < \infty$. When $X = \mathbb{R}$ we shall write $\mathfrak{L}_p(\Omega)$.

Definition 2. The equation (3) – (4) is called \mathfrak{L}_p -equivalent to the equation (1) – (2) in the ball B_r if there exists $\rho > 0$ such that for any solution $x(t)$ of (1) – (2) lying in B_r there exists a solution $y(t)$ of (3) – (4) lying in the ball $B_{r+\rho}$ and satisfying the relation $y(t) - x(t) \in \mathfrak{L}_p(\mathbb{R}_+, X)$. If equation (3) – (4) is \mathfrak{L}_p -equivalent to equation (1) – (2) in the ball B_r and vice versa, we shall say that equations (1) – (2) and (3) – (4) are \mathfrak{L}_p -equivalent in the ball B_r .

3. Main results

3.1. Equivalent equations

Let

$$w(t, s) = \prod_{j=n(t)}^{n(s)+1} Q_j \quad (0 \leq s < t) \quad (5)$$

and

$$\tilde{w}(t, s) = \prod_{i=n(t)}^{n(s)+1} D_i \quad (0 \leq s < t), \quad (6)$$

where $n(\tau) = \max\{n : t_n < \tau\}$.

Lemma 1. *Each solution $x(t)$ of equation (1) – (2) which lies in the ball B_r is a solution of the nonlinear integral equation*

$$x(t) = w(t, 0)x(0) + \sum_{i=0}^{n(t)-1} w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds + \int_{t_{n(t)}}^t F(s, x(s))ds \quad (7)$$

and each solution $y(t)$ of equation (3) – (4) which lies in the ball $B_{r+\rho}$ is a solution of the nonlinear integral equation

$$y(t) = \tilde{w}(t, 0)y(0) + \sum_{i=0}^{n(t)-1} \tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, y(s))ds + \int_{t_{n(t)}}^t G(s, y(s))ds \quad (8)$$

Lemma 1 is proved by straightforward verification.

Set

$$z(t) = y(t) - x(t). \quad (9)$$

Then the function $z(t)$ is a solution of the nonlinear integral equation

$$\begin{aligned} z(t) &= \tilde{w}(t, 0)y(0) - w(t, 0)x(0) + \\ &+ \sum_{i=0}^{n(t)-1} \{ \tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds \} + \\ &+ \int_{t_{n(t)}}^t \{ G(s, x(s) + z(s)) - F(s, x(s)) \} ds. \end{aligned} \quad (10)$$

Let

$$\begin{aligned} H(x, z)(t) &= \tilde{w}(t, 0)y(0) - w(t, 0)x(0) + \\ &+ \sum_{i=0}^{n(t)-1} \{ \tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds \} + \\ &+ \int_{t_{n(t)}}^t \{ G(s, x(s) + z(s)) - F(s, x(s)) \} ds. \end{aligned} \quad (11)$$

Then

$$z(t) = H(x, z)(t). \quad (12)$$

From Definition 2 it follows that to establish the \mathfrak{L}_p - equivalence of equation (3) – (4) to equation (1) – (2) it suffices to show that for each solution $x(t)$ of equation (1) – (2) lying in the ball B_r the operator equation (12) has a fixed point $z(t)$ such that $x(t) + z(t) \in B_{r+\rho}$ for some $\rho > 0$ and which lies in $\mathfrak{L}_p(\mathbb{R}_+, X)$.

Let $S(\mathbb{R}_+, X)$ be the space of all functions which are continuous for $t \neq t_n$ ($n = 1, 2, \dots$), have at the points t_n limits on the left and right and are left continuous. The space $S(\mathbb{R}_+, X)$ is linear and locally convex. A metric can be introduced by

$$\rho(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\max_{t \in (t_n, t_{n+1}]} \|x(t) - y(t)\|}{1 + \max_{t \in (t_n, t_{n+1}]} \|x(t) - y(t)\|}.$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli's theorem is valid.

Lemma 2. [2] *The set $M \subset S(\mathbb{R}_+, X)$ is relatively compact if and only if M is equicontinuous on each interval $(t_{n-1}, t_n]$ ($n = 1, 2, \dots$).*

Proof. We apply the theorem of Arzella-Ascoli on each interval $(t_{n-1}, t_n]$ ($n = 1, 2, \dots$) and constitute diagonal line sequence, which is converging on each from them. \square

In the proof of the existence of a fixed point of the operator H from the equation (12) we use a modification of Schauder's classical principle.

Lemma 3. [2] *Let the operator H transform the set*

$$C(r) = \{x \in S(\mathbb{R}_+, X) : x(t) \in B_r, t \geq 0\}$$

into itself and be continuous and compact.

Then H has a fixed point in $C(r)$.

3.2. Conditions for \mathfrak{L}_p -equivalence

Theorem 1. *Let the following conditions are fulfilled.*

1. *The operator-valued functions $w(t, s)$ and $\tilde{w}(t, s)$ satisfy the condition*

$$\|\tilde{w}(t, 0)\xi - w(t, 0)\eta\| \leq \chi_{r,\rho}(t) \quad (0 \leq t < \infty), \quad (13)$$

where $\xi \in B_{r+\rho}$, $\eta \in B_r$, $\chi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$ and $r, \rho > 0$.

2. *The functions $F(t, x)$, $G(t, y)$ and $w(t, s)$, $\tilde{w}(t, s)$ satisfy the condition*

$$\sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \sum_{i=0}^{n(t)-1} \|\tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, v) ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, u) ds\| \leq \psi_{r,\rho}(t), \quad (14)$$

where $\psi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$ and

$$\sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \int_{t_n(t)}^t \|G(s, v) - F(s, u)\| ds \leq \varphi_{r,\rho}(t), \quad (15)$$

where $\varphi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

3. The function $G(t, y)$ satisfies the condition

$$\sup_{\|v\| \leq r+\rho} \|G(t, v)\| \leq \Phi_{r,\rho}(t), \quad (16)$$

where $\Phi_{r,\rho}(t)$ is integrable on each interval $(t_{n-1}, t_n]$ ($n = 1, 2, \dots$).

4. The inequality

$$\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \leq \rho \quad (17)$$

holds for each $t \geq 0$.

Then the equation (3) – (4) is \mathfrak{L}_p -equivalent to the equation (1) – (2) in the ball B_r .

Proof. We shall show that for any function $x(t) \in B_r$ ($t \geq 0$) the operator $H(x, z)$ defined by (11) maps the set

$$C(\rho) = \{z \in S(\mathbb{R}_+, X) : z(t) \in B_\rho, t \geq 0\}$$

into itself.

Let $x(t) \in B_r$ ($t \geq 0$) and let $z \in C(\rho)$. Then from (11) we obtain the estimate:

$$\begin{aligned} \|H(x, z)(t)\| &\leq \|\tilde{w}(t, 0)y(0) - w(t, 0)x(0)\| + \\ &+ \sum_{i=0}^{n(t)-1} \|\tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s)) ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s)) ds\| + \\ &+ \int_{t_n(t)}^t \|G(s, x(s) + z(s)) - F(s, x(s))\| ds \leq \|\tilde{w}(t, 0)y(0) - w(t, 0)x(0)\| + \\ &+ \sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \sum_{i=0}^{n(t)-1} \|\tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, v) ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, u) ds\| + \\ &+ \sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \int_{t_n(t)}^t \|G(s, v) - F(s, u)\| ds \leq \chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \leq \rho \end{aligned}$$

for each $t \geq 0$.

We obtain $\|H(x, z)(t)\| \leq \rho$, i.e., $H(x, z) \in C(\rho)$. Hence, for any $x \in C(\rho)$, the set $C(\rho)$ is invariant with respect to $H(x, z)$.

Let be $L = \{u(t) = H(x, z)(t) : \|z\| \leq \rho\}$.

First we shall establish that the set L is compact in $S(\mathbb{R}_+, X)$.

We shall show the equicontinuity of the functions of the set L . In fact, for $t', t'' \in (t_{n-1}, t_n]$ following equalities hold:

$$w(t', s) = w(t'', s) = w(t_n, s)$$

$$\tilde{w}(t', s) = \tilde{w}(t'', s) = \tilde{w}(t_n, s)$$

$$n(t') = n(t'') = n - 1$$

For $t', t'' \in (t_{n-1}, t_n]$ we obtain

$$\begin{aligned} & \|u(t') - u(t'')\| = \\ & = \|(\tilde{w}(t', 0)y(0) - w(t', 0)x(0)) - (\tilde{w}(t'', 0)y(0) - w(t'', 0)x(0)) + \\ & + \sum_{i=0}^{n(t')-1} \{\tilde{w}(t', t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t', t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds\}ds - \\ & - \sum_{i=0}^{n(t'')-1} \{\tilde{w}(t'', t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t'', t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds\} + \\ & + \int_{t_{n(t')}}^{t'} \{G(s, x(s) + z(s)) - F(s, x(s))\}ds - \\ & - \int_{t_{n(t'')}}^{t''} \{G(s, x(s) + z(s)) - F(s, x(s))\}ds\| \leq \\ & \leq \left| \sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \int_{t''}^{t'} \|G(s, v) - F(s, u)\|ds \right| \end{aligned}$$

The equicontinuity of the functions of the set L follows from the last estimate.

From Lemma 2 the compactness of the set L follows.

We shall show that the operator $H(x, z)$ is continuous in $S(\mathbb{R}_+, X)$.

Let the sequence $\{z_n(t)\} \subset C(\rho)$ be convergent in the metric of the space $S(\mathbb{R}_+, X)$ (i.e., uniformly converges on each bounded interval) to the function $z(t) \in C(\rho)$. Then, for $t \in \mathbb{R}_+$ the sequence $G(t, x(t) + z_n(t))$ converges to $G(t, x(t) + z(t))$. From conditions 3 of Theorem 1 it follows that the convergent sequence of functions $G(t, x(t) + z_n(t))$ is majorized by the intergrable function

$\Phi_{r,\rho}(t)$. That's why within the integral in formula

$$\begin{aligned} H(x, z_n)(t) &= \tilde{w}(t, 0)y(0) - w(t, 0)x(0) + \\ &+ \sum_{i=0}^{n(t)-1} \left\{ \tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z_n(s))ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds \right\} + \\ &+ \int_{t_{n(t)}}^t \{G(s, x(s) + z_n(s)) - F(s, x(s))\}ds \end{aligned}$$

we may pass to the limit. Hence $H(x, z_n)(t)$ tends to $H(x, z)(t)$ for $t \in \mathbb{R}_+$. Since $H(x, z)$ maps $C(\rho)$ into a compact set, $H(x, z_n)$ tends to $H(x, z)$ in $S(\mathbb{R}_+, X)$ as well.

From Lemma 3 it follows that for any $x \in C(\rho)$ the operator $H(x, z)$ has a fixed point z in $C(\rho)$, i.e. $z = H(x, z)$.

We shall show that this fixed point $z(t)$ lies in $\mathfrak{L}_p(\mathbb{R}_+, X)$.

$$\begin{aligned} \|z(t)\| &\leq \|\tilde{w}(t, 0)y(0) - w(t, 0)x(0)\| + \\ &+ \sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \sum_{i=0}^{n(t)-1} \left\| \tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, v)ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, u)ds \right\| + \\ &+ \sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \int_{t_{n(t)}}^t \|G(s, v) - F(s, u)\|ds \leq \chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \\ \|z\|_p &= \left(\int_0^\infty \|z(t)\|^p dt \right)^{\frac{1}{p}} \leq \left(\int_0^\infty |\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t)|^p dt \right)^{\frac{1}{p}} \leq \\ &\leq \|\chi_{r,\rho}\|_p + \|\psi_{r,\rho}\|_p + \|\varphi_{r,\rho}\|_p \end{aligned}$$

Hence this fixed point belongs to the space $\mathfrak{L}_p(\mathbb{R}_+, X)$, i.e., the equations (3) – (4) are \mathfrak{L}_p -equivalent to the equations (1) – (2) in ball B_r .

Theorem 1 is proved. \square

Remark 1. Condition (13) means that the “impulse difference” of the two equations belongs in the space $\mathfrak{L}_p(\mathbb{R}_+)$.

Condition (14) means that the sum of the “integral differences” of G and F with weights \tilde{w} and w on the balls $B_{r+\rho}$ and B_r respectively on any interval $[t_i, t_{i+1}]$ lies in the space $\mathfrak{L}_p(\mathbb{R}_+)$.

Condition (15) means that the “integral difference” of the ordinary parts on any interval $[t_i, t_{i+1}]$ lies in the space $\mathfrak{L}_p(\mathbb{R}_+)$.

Remark 2. It may be noted that the condition (18) in [2] is not fulfilled if one of the equations is an ordinary. Let the equation (3) – (4) be ordinary i.e. $D_n = I$. Then for any solution of the impulse equation there exists a solution

of the ordinary equation. If we have evidently or numerical representation of the solution of the ordinary equation, then the solution of the impulse equation will be \mathfrak{L}_p -near to this solution.

Corollary 1. *Let the operators Q_n, D_n ($n = 1, 2, \dots$) are linear and the following conditions are fulfilled.*

1. *The operator-valued function $w(t, s)$ and $\tilde{w}(t, s)$ satisfy the conditions*

$$\|\tilde{w}(t, s)\| \leq M \quad (0 \leq s < t < \infty), \quad (18)$$

where M is a positive number and

$$\|\tilde{w}(t, 0)\xi - w(t, 0)\eta\| \leq \chi_{r,\rho}(t) \quad (0 \leq t < \infty), \quad (19)$$

where $\xi \in B_{r+\rho}, \eta \in B_r, \chi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$ and $r, \rho > 0$.

2. *The functions $F(t, x)$ and $G(s, y)$ satisfy the condition*

$$\sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \int_0^t \|\tilde{w}(t, s)G(s, v) - w(t, s)F(s, u)\| ds \leq \psi_{r,\rho}(t), \quad (20)$$

where $\psi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

3. *The function $G(t, y)$ satisfies the condition*

$$\sup_{\|v\| \leq r+\rho} \|G(t, v)\| \leq \varphi_{r,\rho}(t) \in \mathfrak{L}_1(\mathbb{R}_+) \quad (21)$$

4. *The inequality*

$$\chi_{r,\rho}(t) + \psi_{r,\rho}(t) \leq \rho \quad (22)$$

holds for each $t \geq 0$.

Then the equation (3) – (4) is \mathfrak{L}_p -equivalent to the equation (1) – (2) in the ball B_r .

Proof. The corollary follows immediately from the relations

$$x(t) = \tilde{w}(t, 0)x(0) + \int_0^t \tilde{w}(t, s)F(s, x(s))ds,$$

$$y(t) = w(t, 0)y(0) + \int_0^t w(t, s)G(s, y(s))ds. \quad \square$$

Example. Consider the impulse equations

$$\frac{dx}{dt} = F(t, x) \quad \text{for } t \neq n \quad (23)$$

$$x(n^+) = 5^{-n}(2 - \sin x(n)) \quad \text{for } n = 1, 2, \dots \quad (24)$$

and

$$\frac{dy}{dt} = G(t, y) \quad \text{for } t \neq n \quad (25)$$

$$y(n^+) = 5^{-n} \sin y(n) \quad \text{for } n = 1, 2, \dots, \quad (26)$$

where $F(t, x), G(t, y) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Let for some $0 < r < \Delta$ the functions $G(t, y)$ and $F(t, x)$ satisfy the conditions

$$\sup_{\substack{|u| \leq r \\ |v| \leq \Delta}} \int_{[t]}^t |G(s, v) - F(s, u)| ds \leq \varphi(t) \in \mathfrak{L}_p(\mathbb{R}_+) \quad (27)$$

$$\sup_{|v| \leq \Delta} |G(t, v)| \leq \Phi(t) \quad (28)$$

The function $\varphi(t)$ satisfies the condition

$$4.5^{-[t]} + 4.t.5^{-[t]} + \varphi(t) \leq \Delta - r \quad (29)$$

The function $\Phi(t)$ is integrable on each interval $(n-1, n]$ ($n = 1, 2, \dots$).

We note that the conditions (27) – (29) are fulfilled for example by

$$F(t, x) = \frac{\ln 5}{4} 5^{-t} \frac{x}{1+x^2}$$

$$G(t, y) = \frac{\ln 5}{4} 5^{-t} y \sin^2 y$$

Indeed in this case we have

$$\begin{aligned} \sup_{\substack{|u| \leq r \\ |v| \leq \Delta}} \int_{[t]}^t \frac{\ln 5}{4} 5^{-s} \left| v \sin^2 v - \frac{u}{1+u^2} \right| ds &= \frac{1}{4} (5^{-[t]} - 5^{-t}) \sup_{\substack{|u| \leq r \\ |v| \leq \Delta}} \left| v \sin^2 v - \frac{u}{1+u^2} \right| \leq \\ &\leq \frac{1}{2} 5^{-[t]} \left(\sup_{|v| \leq \Delta} |v \sin^2 v| + \sup_{|u| \leq r} \frac{|u|}{1+u^2} \right) \leq \frac{1}{2} 5^{-[t]} (\Delta + r) \end{aligned}$$

Set

$$\varphi(t) = \frac{1}{2} 5^{-[t]} (\Delta + r) \in \mathfrak{L}_p(\mathbb{R}_+)$$

The function $\varphi(t)$ satisfies (29)

$$4.5^{-[t]} + 4.t.5^{-[t]} + \frac{1}{2}5^{-[t]}(\Delta + r) < \Delta - r,$$

for each $t \geq 0$.

Otherwise

$$\sup_{|v| \leq \Delta} |G(t, v)| = \sup_{|v| \leq \Delta} \left| \frac{\ln 5}{4} 5^{-t} v \sin^2 v \right| \leq \frac{\ln 5}{4} \Delta 5^{-t} \in \mathfrak{L}_p(\mathbb{R}_+)$$

We shall show that the conditions of Theorem 1 are fulfilled.

We have

$$Q_n x = 5^{-n}(2 - \sin x), D_n y = 5^{-n} \sin y.$$

Then for any $\xi \in B_\Delta$, $\eta \in B_r$ ($0 < r < \Delta$), $t \in (t_n, t_{n+1}]$ we obtain

$$\begin{aligned} |\tilde{w}(t, 0)\xi - w(t, 0)\eta| &= \left| \prod_{i=[t]}^1 D_i \xi - \prod_{i=[t]}^1 Q_i \eta \right| = \\ &= |D_n \xi_{n-1} - Q_n \eta_{n-1}| = |5^{-[t]} \sin \xi_{n-1} - 5^{-[t]}(2 - \sin \eta_{n-1})| \leq 4.5^{-[t]}, \end{aligned}$$

where

$$\xi_{n-1} = D_{n-1} D_{n-2} \dots D_1 \xi, \quad \eta_{n-1} = Q_{n-1} Q_{n-2} \dots Q_1 \eta$$

Set $\chi(t) = 4.5^{-[t]}$.

We shall show that $\chi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$

$$\int_0^\infty |\chi_{r,p}(t)|^p dt = \int_0^\infty |4.5^{-[t]}|^p dt = 4^p \int_0^\infty 5^{(1-t)p} dt = 20^p \int_0^\infty 5^{-pt} dt < \infty$$

Hence $\chi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

We shall show that the condition 2 of Theorem 1 is fulfilled. Let $t \in (t_n, t_{n+1}]$. Then

$$\begin{aligned} &\sup_{\substack{|u| \leq r \\ |v| \leq \Delta}} \left| \sum_{i=0}^{[t]-1} |\tilde{w}(t, i^+) \int_i^{i+1} G(s, v) ds - w(t, i^+) \int_i^{i+1} F(s, u) ds| \right| = \\ &= \sup_{\substack{|u| \leq r \\ |v| \leq \Delta}} \left| \sum_{i=0}^{[t]-1} |5^{-[t]} \sin \xi_{n-1, i+1} - 5^{-[t]}(2 - \sin \eta_{n-1, i+1})| \right| \leq \\ &\leq \sum_{i=0}^{[t]-1} 4.5^{-[t]} \leq 4.t.5^{-[t]}, \end{aligned}$$

where

$$\begin{aligned} \xi_{n-1, i+1} &= D_{n-1} D_{n-2} \dots D_{i+1} \int_i^{i+1} G(s, v) ds \\ \eta_{n-1, i+1} &= Q_{n-1} Q_{n-2} \dots Q_{i+1} \int_i^{i+1} F(s, u) ds \end{aligned}$$

Set $\psi(t) = 4.t.5^{-[t]}$.

We shall show that $\psi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

$$\begin{aligned} \int_0^\infty |\psi(t)|^p dt &= \int_0^\infty |4.t.5^{-[t]}|^p dt = 4^p \int_0^\infty t^p .5^{-p[t]} dt \leq \\ &\leq 4^p \int_0^\infty t^p .5^{(1-t)p} dt = 20^p \int_0^\infty t^p .5^{-pt} dt < \infty. \end{aligned}$$

Hence $\psi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

For the condition 4 of Theorem 1 we obtain

$$\chi(t) + \psi(t) + \varphi(t) = 4.5^{-[t]} + 4.t.5^{-[t]} + \varphi(t) \leq \Delta - r$$

for each $t \geq 0$. Hence the equation (25) – (26) is \mathfrak{L}_p -equivalent to the equation (23) – (24) in the ball B_r ($0 < r < \Delta$).

References

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